TRADE-OFFS IN THE DEBATE FOR SETTING BIOLOGICAL AND PRODUCTIVE THRESHOLDS THAT ARE SUSTAINABLE*

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In this document we propose a methodology in order to compute objectives that can be sustained, under some monotonicity properties, putting in evidence the trade-off between environmental objectives and production objectives. The proposed methodology is illustrated by a case of study on a Chilean fishery.

1. Introduction

Sustainable development issues are dynamics and encompass several conflicting dimensions between environmental and economic/productive objectives. For instance, to harvest a renewable natural resource satisfying environmental constraints versus to obtain minimal revenues. Environmental objectives, or constraints of environmental type to be satisfied, should be recommended by specialists in the biology of the resource, in order, among others, to ensure that resource is not in danger to disappear and/or its capacity of reproduction is not being largely affected. Production objectives, or constraints of production type to be satisfied, can be established by regulatory organisms that pretend to ensure a minimal economic activity in the sector that exploit the resource, inspired by social or productive purposes. Thus, in this framework, there are clearly two groups of interest: environmental group v/s production group. In general, if the environmental constraints are too restrictive, the production objectives (let us think

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in minimal level of harvest) can not be too exigent and viceversa but, in practice, it is not an easy task to model this trade-off.

For some renewable natural resources, in some countries, there are committees whose goal is to recommend (or to impose) environmental and production objectives. For instance, we can think in committees composed by regulatory organisms, stakeholders and scientists for determining quotas of captures (of one fish stock) and minimal levels of spawning stock biomass, or for determining green house gas levels of emission. Once objectives (constraints) are determined by the committees, how to know if it is possible to respect these constraints from a feasibility point of view? Of course the answer should depend on the current state of the resource but, how? A mathematical framework for modeling this problem is the control theory in discrete-time and the associated viability theory.

The goal of this paper is to propose a methodology in order to compute objectives that can be sustained, under some monotonicity properties, and thus, to put in evidence the mentioned trade-off between environmental objectives and production objectives. The proposed methodology will be illustrated by a case of study on a Chilean fishery.

2. Viability in discrete time

Let us consider a nonlinear control system described in discrete time by the difference equation

$$\begin{cases} N(t+1) = g(N(t), u(t)), & t = t_0, t_0+1, \dots \\ N(t_0) \quad \text{given,} \end{cases}$$

where the *state variable* $N(t)$ belongs to the finite dimensional state space $\mathbb{X} = \mathbb{R}^{n_{\mathbb{X}}}$, the *control variable* $u(t)$ is an element of the *control set* $\mathbb{U} = \mathbb{R}^{n_{\mathbb{U}}}$ while the *dynamics* g maps $\mathbb{X} \times \mathbb{U}$ into $\mathbb{X}$. As motivation, we can think that the state N represents the abundances of a renewable natural resource.

A decision maker describes *acceptable configurations of the system* through a set $\mathbb{D} \subset \mathbb{X} \times \mathbb{U}$ termed the *acceptable set*

$$(N(t), u(t)) \in \mathbb{D}\,, \quad \forall t = t_0, t_0+1, \dots$$

where $\mathbb{D}$ includes both system states and controls constraints. Typical instances of such an acceptable set are given by inequalities requirements:

$$\mathbb{D}_\theta = \{(N, u) \in \mathbb{X} \times \mathbb{U} \mid \forall i = 1, \dots, p\,, \quad I_i(N, u) \geq \theta_i\},$$

where the functions $I_1, \dots, I_p$ may be interpreted as *indicators*, and $\theta = (\theta_1, \dots, \theta_p)$ is the vector of corresponding *thresholds*. For management issues, the set $\mathbb{D}_\theta$ will be the mathematical expression of preservation and/or production objectives.

Viability is defined (see [1, 2] and the references therein) as the ability to choose, at each time step $t = t_0, t_0 + 1, \ldots$, a control $u(t) \in \mathbb{U}$ such that the system configuration remains acceptable. More precisely, the system is viable if the following feasible set is not empty:

$$\mathbb{V}(g, \mathbb{D}_\theta) := \left\{ N_0 \in \mathbb{X} \;\middle|\; \begin{array}{l} \exists\ (u(t_0), u(t_0+1), \ldots) \text{ and } (N(t_0), N(t_0+1), \ldots) \\ \text{satisfying } N(t_0) = N_0\,, \quad N(t+1) = g(N(t), u(t)) \\ \text{and } (N(t), u(t)) \in \mathbb{D}_\theta\,, \quad \forall t = t_0, t_0+1, \ldots \end{array} \right\} .$$

For a decision maker, knowing the viability kernel has practical interest since it describes the set of states from which controls can be found that maintain the system in an acceptable configuration forever. However, computing this kernel is not an easy task in general. On the other hand, given an initial state N_0 it will be interesting to determine what are the vectors of threshold θ such that $N_0 \in \mathbb{V}(g, \mathbb{D}_\theta)$.

2.1. *Sustainable thresholds*

The set of sustainable thresholds starting from an initial condition N_0 is given by

$$\mathcal{S}(N_0) := \left\{ \theta = (\theta_1, \ldots, \theta_p) \in \mathbb{R}^p \;\middle|\; \begin{array}{l} \exists\ (u(t_0), u(t_0+1), \ldots) \text{ and} \\ (N(t_0), N(t_0+1), \ldots) \\ \text{satisfying } N(t_0) = N_0 \\ N(t+1) = g\big(N(t), u(t)\big) \\ \forall\ t = t_0, t_0+1, \ldots \text{ and} \\ I_i\big(N(t), u(t)\big) \geq \theta_i \quad \forall\ i = 1, \ldots, p \end{array} \right\} . \tag{1}$$

From the definition of the viability kernel given in the previous section, one has

$$\mathcal{S}(N_0) := \{\theta \in \mathbb{R}^p \mid N_0 \in \mathbb{V}(g, \mathbb{D}_\theta)\}. \tag{2}$$

Notice that if $\theta \in \mathcal{S}(N_0)$ then $\theta' \leq \theta$ (with the componentwise order) belongs also to $\mathcal{S}(N_0)$.

To determine the set of thresholds $\mathcal{S}(N_0)$, for a given initial condition N_0, has practical interests. In fact, if a vector of threshold $\theta = (\theta_1, \theta_2, \ldots, \theta_p)$ represents the minimal objectives to be reached during the harvesting of a natural resources and these objectives are determined in a bargaining process where many interest (environmental and productive) are represented, the goal of this procedure should be to choose a vector θ in $\mathcal{S}(N_0)$.

Under some monotonicity assumptions, to be introduced in the next section, for an initial state N_0 we shall provide methods for:

(1) To compute $\mathcal{S}(N_0)$;

(2) Given $p-1$ thresholds $\theta_{1:p-1} := (\theta_1, \ldots, \theta_{p-1})$ to compute the maximal thresholds θ_p such that $\theta = (\theta_1, \theta_2, \ldots, \theta_p) \in \mathcal{S}(N_0)$.

The second statement is motivated when there is a priority in the determination of some thresholds (θ_p) and therefore the idea is to compute the maximum reachable for the other given thresholds $\theta_{1:p-1} = (\theta_1, \ldots, \theta_{p-1})$. In the management of natural resources, the threshold to be fixed in advance could be such related to environmental constraints and the other related to a productive one.

3. Monotone bioeconomics models

Some dynamic models have the following qualitative properties (*ceteris paribus*): (i) the higher the state vector at a period is, the higher it is at the following period; (ii) the higher the decision at a period is, the lower the state vector is at the following period. As we put a particular focus on environmental issues, let us emphasize that these properties are satisfied, for instance, for problems of natural renewable resource stocks and harvesting[a] or air quality dynamics and pollutant emissions[b].

[a]The larger the resource stock at one period, the larger at the following. The larger the extraction, the lower the resource stock at the following period. Note that these assumptions are not satisfied for multispecies ecological models when there is a prey-predator relationship, as a larger predator stock may reduce the next period prey stock.
[b]The better the air quality at one period, the better at the following period (*ceteris paribus*). And the higher the pollutant emission at one period, the worse the air quality at the following period. This works for the climate change issue and greenhouse gases emissions, taking the negative level of CO_2 atmospheric concentration as a state.

Monotonic indicators and interest groups. With respect to the indicators, we can also exhibit such monotonicity properties. If all all the components of the vector state N are defined as “goods,” indicators will usually increase with the state, i.e., the larger the state vector, the higher the indicators.[c] Some indicators may also be monotonically responding to the decisions. This is the case for environmental indicators which are monotonically decreasing with the decisions such as resource extraction or pollutant emissions.[d]

In order to represent the mentioned above behaviors, we supply the state space $\mathbb{X} \subset \mathbb{R}^{n_{\mathbb{X}}}$ and the decision space $\mathbb{U} \subset \mathbb{R}^{n_{\mathbb{U}}}$ with the componentwise order: $N' \geq N$ if and only if each component of $N' = (N'_1, \ldots, N'_d)$ is greater or equal than to the corresponding component of $N = (N_1, \ldots, N_d)$. We say that a mapping $f : \mathbb{X} \times \mathbb{U} \longrightarrow \mathbb{R}^d$, defined for state and decision variables, with values in $\mathbb{R}^d$ (we will use $d = n_{\mathbb{X}}$ for the dynamics case, and $d = 1$ for the indicator case), is increasing with respect to the state variable if it satisfies $\forall\ (N, N', u) \in \mathbb{X} \times \mathbb{X} \times \mathbb{U}$, $N' \geq N \Rightarrow f(N', u) \geq f(N, u)$, and is *decreasing with respect to the decision* if $\forall\ (N, u, u') \in \mathbb{X} \times \mathbb{U} \times \mathbb{U}$, $u' \geq u \Rightarrow f(N, u') \leq f(N, u)$. Obviously, according to the previous definition, if a function does not depend on the state or the decision, it will be both increasing and decreasing with respect to such variable.

Definition 3.1. Let $k \in \{1, \ldots, p-1\}$. We will say that the dynamics g and the indicators $I_1, \ldots, I_p$ satisfy the $\mathtt{MONDAI}_k$ property if:

- the dynamics $g : \mathbb{X} \times \mathbb{U} \longrightarrow \mathbb{X}$ is increasing in the state variable and decreasing in the decision;
- all the indicators $I_i : \mathbb{X} \times \mathbb{U} \longrightarrow \mathbb{R}$ are continuous, and are increasing in the state variable;
- the first k indicators $I_1, \ldots, I_k$ are decreasing in the decision variable.

In the previous definition, all the indicators are increasing with the state. This assumption means that all the components of the state vari-

[c]This is true for economic indicators, which may depend for instance on capital stocks, knowledge / human capital, or infrastructures. This is also true for ecological indicators as long as the capital stocks are properly defined, by accounting for “bads” (pollution for instance) by their negative level.

[d]Note that economic indicators may be monotonically increasing with the decisions, but not necessarily. For example, fishermen may favor an increase of fishing effort as long as it increases their profit, but no more when the associated cost is higher than the benefit from fishing.

able are valuable (or at least not damageable). The agents represented by indicators of the first group $\{1,\dots,k\}$ have a particular interest in having the decision always as small as possible (e.g., fishing effort, GHG emissions, deforestation), which is interpreted as a "pro-environmental group" in our environmental issue context. Their indicators are always decreasing when the decision variables increase. On the contrary, the second group of indicators does not depend on the decision in a particular way (or some of the indicators may be increasing with the decision, in opposition to the indicators of the first group). The agents represented by indicators of the second group $\{k+1,\dots,p\}$ are called *outsiders* of the interest group $\{1,\dots,k\}$ as they have no systematic "monotonic" interest in the decision level.

When the sequence of actions or controls $u(t_0), u(t_0+1)\dots$ is defined by a feedback rule, i.e., a mapping $\mathfrak{u}:\mathbb{X}\to\mathbb{U}$ giving each decision as a function of the state by $u(t)=\mathfrak{u}\big(N(t)\big)$, one gets the closed-loop dynamics

$$\begin{cases} N(t_0) = N_0 \\ u(t) = \mathfrak{u}\big(N(t)\big) \\ N(t+1) = g\big(N(t), u(t)\big)\,. \end{cases} \tag{3}$$

For a vector of satisficing outcomes $\theta=(\theta_1,\dots,\theta_p)\in\mathcal{S}(N_0)$ – under the monotony assumptions $\texttt{MONDAI}_k$ – we shall describe a common feedback decision rule that ensures to obtain at least these thresholds. This rule will be parametrized by thresholds of the outsiders of the interest group.

In what follows, we will consider a scalar decision, i.e., $\mathbb{U}\subset\mathbb{R}$.

The next result, which is a slight extension of the result established in [5], provides in the $\texttt{MONDAI}_k$ framework a tool for determining whenever a vector of threshold $I=(I_1,\dots,\theta_p)$ belongs to $\mathcal{S}(N_0)$ or not.

Proposition 3.1. *Assume that the dynamics g and the indicators $I_1,\dots,I_p$ satisfy the $\texttt{MONDAI}_k$ property and the control set $\mathbb{U}$ is bounded and closed.*

Consider $p-k$ thresholds $\theta_{k+1:p}=(\theta_{k+1},\dots,\theta_p)\in\mathbb{R}^{p-k}$ and define the decision rule $\mathfrak{u}^\star_{\theta_{k+1:p}}$ by[e]

$$\mathfrak{u}^\star_{\theta_{k+1:p}}(N) := \inf\{u\in\mathbb{U}\mid I_i(N,u)\geq\theta_i\,,\quad i=k+1,\dots,p\}\,. \tag{4}$$

Then, for any $\theta_{1:k}=(\theta_1,\dots,\theta_k)\in\mathbb{R}^k$, the vector of thresholds $\theta=$

[e]Notice that $\mathfrak{u}^\star_{\theta_{k+1:p}}(N)$ is not defined for thoses states N such that $\{u\in\mathbb{U}\mid I_i(N,u)\geq\theta_i\,,\quad i=k+1,\dots,p\}=\emptyset$.

$(\theta_{1:k}, \theta_{k+1:p})$ belongs to $\mathcal{S}(N_0)$ if and only if the feedback decision $\mathfrak{u}^\star_{\theta_{k+1:p}}$ is a common decision rule that allows to obtain at least θ, starting from N_0.

Proof. We have to show that for an initial state N_0, if the vector of thresholds $\theta = (\theta_{1:k}, \theta_{k+1:p})$ belong to $\mathcal{S}(N_0)$ then, $u^\star(t) = \mathfrak{u}^\star_{\theta_{k+1:p}}(N(t))$ defined in (4) is a decision rule that allows to obtain at least θ.

Take $\theta = (\theta_{1:k}, \theta_{k+1:p}) \in \mathcal{S}(N_0)$ and a sequence of decisions $u(t_0), u(t_0 + 1) \ldots$ that allows to guarante these thresholds. Since $\theta \in \mathcal{S}(N_0)$, the decision $\mathfrak{u}^\star_{\theta_{k+1:p}}(N_0)$ is well defined (the infimum is taken over an nonempty set) and (from the definition of $\mathfrak{u}^\star_{\theta_{k+1:p}}(\cdot)$ in (4)), we have that $u(t_0) \geq \mathfrak{u}^\star_{\theta_{k+1:p}}(N(t_0))$ and therefore, due to g is decreasing in the decision variable, we obtain that

$$N^\star(t_0+1) = g(N(t_0), \mathfrak{u}^\star_{\theta_{k+1:p}}(N(t_0))) \geq g(N(t_0), u(t_0)) = N(t_0+1) \; .$$

In the following we will denote by $N^\star(\cdot)$ and $N(\cdot)$ the trajectories of the states generated by feedback decisions $\mathfrak{u}^\star_{\theta_{k+1:p}}$ and decisions $u(\cdot)$ respectively.

Since indicators I_i, $i = k+1, \ldots, p$, are increasing with the state N, we can see in (4) that $\mathfrak{u}^\star_{\theta_{k+1:p}}(N)$ is decreasing with the state N. Hence

$$\begin{aligned} u(t_0+1) &\geq \mathfrak{u}^\star_{\theta_{k+1:p}}(N(t_0+1)) \\ &\qquad \text{by definition of } \mathfrak{u}^\star_{\theta_{k+1:p}} \text{ because} \\ &\qquad I_i(N(t_0+1), u(t_0+1)) \geq \theta_i \;, \quad i = k+1, \ldots, p \\ &\geq \mathfrak{u}^\star_{\theta_{k+1:p}}(N^\star(t_0+1)) \\ &\qquad \text{because } \mathfrak{u}^\star_{\theta_{k+1:p}}(\cdot) \text{ is decreasing in the state variable .} \end{aligned}$$

We thus obtain that

$$\begin{aligned} N^\star(t_0+2) &= g(N^\star(t_0+1), \mathfrak{u}^\star_{\theta_{k+1:p}}(N^\star(t_0+1))) \\ &\geq g(N^\star(t_0+1), u(t_0+1)) \\ &\quad \text{because the dynamics } g \text{ is decreasing in the control variable} \\ &\geq g(N(t_0+1), u(t_0+1)) \\ &\quad \text{because the dynamics } g \text{ is increasing in the state variable} \\ &= N(t_0+2) \; . \end{aligned}$$

Recursively we can conclude that $N^\star(t) \geq N(t)$ and $\mathfrak{u}^\star_{\theta_{k+1:p}}(N^\star(t)) \leq u(t)$ for all $t \geq t_0$.

On the other hand, by assumption, the indicators I_1, ..., I_k are increasing in the state and decreasing in the decision variable. We deduce then that for $i = 1, \ldots, k$,

$$I_i\big(N^\star(t), \mathfrak{u}^\star_{\theta_{k+1:p}}(N^\star(t))\big) \geq I_i\big(N(t), u(t)\big) \geq \theta_i \ .$$

For $i = k+1, \ldots, p$, notice that $I_i\big(N^\star(t), \mathfrak{u}^\star_{\theta_{k+1:p}}(N^\star(t))\big) \geq \theta_i$ by definition of $\mathfrak{u}^\star_{\theta_{k+1:p}}$ which allows to conclude the desired result.

Finally, if $\mathfrak{u}^\star_{\theta_{k+1:p}}$ is a common decision rule that allow to obtain at least $\theta = (\theta_{1:k}, \theta_{k+1:p})$, obviously $\theta \in \mathcal{S}(N_0)$. □

The interest of the previous result is twofold. On the one hand, if $(\theta_1, \ldots, \theta_p) \in \mathcal{S}(N_0)$, the trajectory starting from the initial state N_0 and defined by the feedback rule $\mathfrak{u}^\star_{\theta_{k+1:p}}$, that is

$$\begin{cases} N(t_0) = N_0 \\ N(t+1) = g\left(N(t), \mathfrak{u}^\star_{\theta_{k+1:p}}(N(t))\right) \quad t = t_0, t_0+1, \ldots, \end{cases} \tag{5}$$

guarantee to satisfy the constraints for the thresholds $\theta_1, \ldots, \theta_p$. On the other hand, given a partial set of outcomes $\theta_{k+1:p}$, if the economic trajectory (5) defined by $\mathfrak{u}^\star_{\theta_{k+1:p}}$ does not achieve a given complementary set of outcomes $\tilde{\theta}_{1:k}$, no other rule will. It means that outcomes $(\tilde{\theta}_{1:k}, \theta_{k+1:p})$ cannot be guaranteed, namely $(\tilde{\theta}_{1:k}, \theta_{k+1:p}) \notin \mathcal{S}(N_0)$.

Maximal minimal economic outcome Notice that in the particular case $\texttt{MONDAI}_{p-1}$, that is, the last indicator is an economic instantaneous payoff and the first $p-1$ indicators are of an environmental type, one may consider the economic problem of maximizing the economic payoff under environmental constraints (i.e. given the environmental thresholds).

Proposition 3.1 provides us a tool in order to compute, starting from a state N_0, the maximal threshold reachable as minimal outcome for the last indicator, when the other thresholds are fixed. Indeed, under $\texttt{MONDAI}_{p-1}$ assumptions, for a fixed set of thresholds $\theta_{1:p-1} = (\theta_1, \ldots, \theta_{p-1})$ and different threshold θ_p, we can run the dynamics

$$\begin{cases} N(t_0) = N_0 \\ N(t+1) = g\big(N(t), \mathfrak{u}^\star_{\theta_p}(N(t))\big) \quad t = t_0, t_0+1, \ldots, \end{cases} \tag{6}$$

where $\mathfrak{u}^\star_{\theta_p}(N(t)) = \min\{u \in \mathbb{U} \mid I_p(N(t), u) \geq \theta_p\}$. If dynamics (6) is well defined, that is, if for all t the set $\{u \in \mathbb{U} \mid I_p(N(t), u) \geq \theta_p\}$ is not empty, then, from Proposition 3.1 we obtain that $\theta = (\theta_1, \theta_2, \ldots, \theta_p)$ belongs to $\mathcal{S}(N_0)$. In such a case, we can increase θ_p and run again the dynamics (6) with the associated decision rule $\mathfrak{u}^{\star}_{\theta_p}(N(\cdot))$. On the other hand, if (6) is not well defined, we try with a lower θ_p. Thus, recursively we can construct a sequence of thresholds θ_p converging to

$$\theta^{\star}_p(\theta_{1:p-1}, N_0) = \max\{\theta_p \ \mid \ \theta = (\theta_1, \ldots, \theta_p) \in \mathcal{S}(N_0)\}. \tag{7}$$

We state in Algorithm 1 a simple bisection procedure to compute the value $\theta^{\star}_p(\theta_{1:p-1}, N_0)$.

Algorithm 1 Computation of maximum value $\theta^{\star}_p(\theta_{1:p-1}, N_0)$

Require: Initial state N_0, thresholds $\theta_{1:p-1} = (\theta_1, ..., \theta_{p-1})$, maximal time $T_{max} > t_0$ and a tolerance ε.

$\theta_p^{min} := \min_{u \in \mathbb{U}} I_p(N_0, u)$
$\theta_p^{max} := \max_{u \in \mathbb{U}} I_p(N_0, u)$
while $\theta_p^{max} - \theta_p^{min} \geq \varepsilon$ **do**
    $N(t_0) = N_0$
    $\mathit{notviable} := 0$
    $t := t_0$
    $\theta_p = (\theta_p^{max} + \theta_p^{min})/2$
    while $t \leq T_{max}$ and $\mathit{notviable} = 0$ **do**
        if $\{u \in \mathbb{U}\,;\, I_p(N(t), u) \geq \theta_p\} \neq \emptyset$ **then**
            $\bar{u} := \min\{u \in \mathbb{U}\,;\, I_p(N(t), u) \geq \theta_p\}$
            if $I_i(N(t)), \bar{u}) \geq \theta_i,\ \forall i = 1, ..., p-1$ **then**
                $N(t+1) := g(N(t), \bar{u})$
            else
                $\theta_p^{max} := \theta_p$
                $\mathit{notviable} := 1$
            end if
            $t := t + 1$
        else
            $\mathit{notviable} := 1$
        end if
    end while
    if $\mathit{notviable} = 0$ **then**
        $\theta_p^{min} = \theta_p$
    end if
end while

Ensure: $\theta_p := \theta_p^{min}$

The boolean variable *notviable* defined in line 5 indicates if the current value of θ_p allows that $N_0 \in \mathbb{V}(g, \mathbb{D}_\theta)$ for the given reference points values $\theta_{1:p-1}$, where $\theta = (\theta_{1:p-1}, \theta_p)$. So, variable *notviable* became 1 in line 15 iff $N(t) \notin \mathbb{V}(g, \mathbb{D}_\theta)$ for the tested values, and then our algorithm stops the inner while cycle defined between lines 8 and 21. In such case a smaller value of θ_p is tested in the next iteration. Otherwise, the inner while cycle ends after T_{max} iterations and a lager value of θ_p is tested in the next iteration (see if sentence in lines 22–24). Hence, the algorithm finishes obtaining the largest value of θ_p such that $\theta = (\theta_1, \ldots, \theta_p) \in \mathcal{S}(N_0)$ or, equivalently, $N_0 \in \mathbb{V}(g, \mathbb{D}_\theta)$.

The main computation of the algorithm is done in line 10. This is a one variable optimization problem, which can be solved very easily for some particular constraint I_p. For instance, in many management problem, function $I_p(N, \cdot)$ is monotone for any N. Then, in this case, line 10 is replaced by solving the equation $I_p(N(t), u) = \theta_p$.

The maximal threshold θ_p to be considered at the beginning of the algorithm can be also computed using the methodology introduced in [4].

3.1. *Non-cooperative sustainable thresholds*

Thanks to the result of Proposition 3.1, we shall provide a way to describe the set of sustainable thresholds $\mathcal{S}(N_0)$.

Proposition 3.2. *If the dynamics g and the indicators $I_1, \ldots, I_p$ are* `MONDAI`$_k$ *for some $k \in \{1, \ldots, p-1\}$ then,*

$$\mathcal{S}(N_0) = \{\theta = (\theta_{1:k}, \theta_{k+1:p}) \in \mathbb{R}^p \ \mid \theta_{1:k} \le \Theta_{1:k}(\theta_{k+1:p}, N_0)\} \tag{8}$$

where the components of $\Theta_{1:k}(\theta_{k+1:p}, N_0) = (\Theta_1(\theta_{k+1:p}, N_0), \ldots, \Theta_k(\theta_{k+1:p}, N_0))$ *are defined by*

$$\Theta_i(\theta_{k+1:p}, N_0) = \inf_{t=t_0,t_0+1,\ldots} I_i\big(N(t), \mathfrak{u}^\star_{\theta_{k+1:p}}(N(t))\big) \qquad i = 1, \ldots, k\ . \tag{9}$$

Here above, the decision rule $\mathfrak{u}^\star_{\theta_{k+1:p}}$ is given by (4) *and the state by the closed loop dynamics* (5)*.*

Proof.

For $\theta = (\theta_1, \ldots, \theta_p) = (\theta_{1:k}, \theta_{k+1:p})$ in $\mathcal{S}(N_0)$ we first prove that the inequalities $\theta_i \le \Theta_i(\theta_{k+1:p}, N_0)$ for $i = 1, \ldots, k$ hold. From the definition of $\mathcal{S}(N_0)$, there exists a sequence of controls $u(t_0), u(t_0+1), \ldots$ such that the trajectory given by

$$\begin{cases} \tilde{N}(t+1) = g(\tilde{N}(t), u(t)), \quad t = t_0, t_0+1, \dots \\ \tilde{N}(t_0) = N_0 \end{cases}$$

satisfies

$$I_i(\tilde{N}(t), u(t)) \geq \theta_i \qquad i = 1, 2, \dots, p \qquad t = t_0, t_0+1, \dots . \tag{10}$$

Since $I_i(N_0, u(t_0)) \geq \theta_i$, for $i = k+1, \dots, p$, from the definition of $\mathfrak{u}^{\star}_{\theta_{k+1:p}}$ one has $u(t_0) \geq \mathfrak{u}^{\star}_{\theta_{k+1:p}}(N_0)$ which, from (10) (for $t = t_0$) and monotonicity properties of indicators $I_1, \dots, I_k$, implies

$$I_i\big(N_0, \mathfrak{u}^{\star}_{\theta_{k+1:p}}(N_0)\big) \geq \theta_i \qquad i = 1, \dots, k \ .$$

If we consider the trajectory

$$\begin{cases} N(t+1) = g\big(N(t), \mathfrak{u}^{\star}_{\theta_{k+1:p}}(N(t))\big), \quad t = t_0, t_0+1, \dots \\ N(t_0) = N_0 \end{cases} \tag{11}$$

inductively we can prove that $\mathfrak{u}^{\star}_{\theta_{k+1:p}}(N(t)) \leq u(t)$ and $N(t) \geq \tilde{N}(t)$ for all $t = t_0, t_0+1, \dots$. Therefore

$$I_i\big(N(t), \mathfrak{u}^{\star}_{\theta_{k+1:p}}(N(t))\big) \geq I_i\big(\tilde{N}(t), u(t)\big) \geq \theta_i \qquad i = 1, \dots, k, \ t = t_0, t_0+1, \dots$$

implying $\theta_i \leq \Theta_i(\theta_{k+1:p}, N_0)$ for $i = 1, \dots, k$.

For the reverse inclusion in (8), take $\theta = (\theta_{1:k}, \theta_{k+1:p}) \in \mathbb{R}^p$. If $\theta_{1:k} \leq \Theta_{1:k}(\theta_{k+1:p}, N_0)$, from he definition of $\Theta_i(\theta_{k+1:p}, N_0)$ in (9), we have that the trajectory defined in (11) satisfies

$$I_i\big(N(t), \mathfrak{u}^{\star}_{\theta_{k+1:p}}(N(t))\big) \geq \Theta_i(\theta_{k+1:p}, N_0) \geq \theta_i \qquad i = 1, \dots, k, \quad t \geq t_0$$

and, from definition of $\mathfrak{u}^{\star}_{\theta_{k+1:p}}(\cdot)$, one has

$$I_i\big(N(t), \mathfrak{u}^{\star}_{\theta_{k+1:p}}(N(t))\big) \geq \theta_i \qquad i = k+1, \dots, p, \ t \geq t_0$$

concluding that $\theta = (\theta_{1:k}, \theta_{k+1:p}) \in \mathcal{S}(N_0)$, because the common decision rule $\mathfrak{u}^{\star}_{\theta_{k+1:p}}$ is admissible for θ. □

Equality (8) establishes that the set of satisficing thresholds is parameterized by the $p-k$ thresholds associated to the outsiders group. Indeed, the outcomes $\theta = (\Theta_{1:k}(\theta_{k+1:p}, N_0), \theta_{k+1:p})$, when $\theta_{k+1:p}$ covers different values on $\mathbb{R}^{p-k}$, allow to compute the set $\mathcal{S}(N_0)$ by the relation (deduced from Proposition 3.2)

$$\mathcal{S}(N_0) = \widehat{\mathcal{S}}(N_0) + \mathbb{R}^p_-$$

where

$$\widehat{\mathcal{S}}(N_0) = \left\{(\Theta_{1:k}(\theta_{k+1:p}, N_0), \theta_{k+1:p}) \;\mid\; \theta_{k+1:p} \in \mathbb{R}^{p-k}\right\} \ , \qquad (12)$$

and $\mathbb{R}^p_-$ is the p dimensional negative octant $\mathbb{R}^p_- = \{(\sigma_1, \ldots, \sigma_p) \mid \sigma_i \leq 0\,, \quad i = 1, \ldots, p\}$. Thus, the set of satisficing outcomes $\mathcal{S}(N_0)$ is obtained by means of $\widehat{\mathcal{S}}(N_0)$ which is more tractable to compute.

Figure 1 illustrates how to compute $\widehat{\mathcal{S}}(N_0)$ and therefore $\mathcal{S}(N_0)$. The figure corresponds to a case with $p = 3$ and $k = 2$. Taking $\mathfrak{u}^\star_{\theta_3}(N) = \inf\{u | I_3(N, u) \geq \theta_3\}$ and computing Θ_1 and Θ_2 for all θ_3.

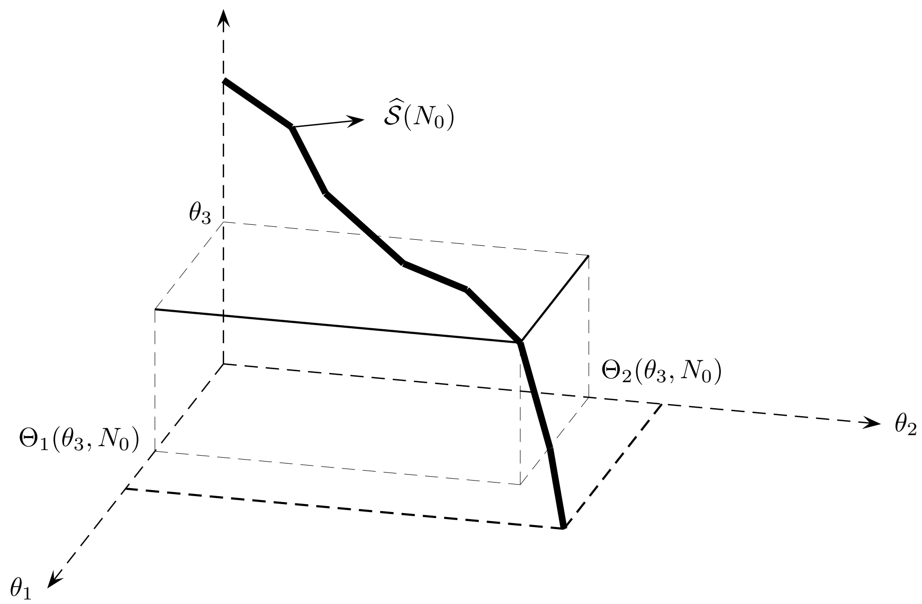

Figure 1. Sustainable thresholds parameterized by threshold θ_3.

4. Application to fishery management

In this section we apply and specify the previous results in the case of an age-structured abundance population model, especially with a Beverton-Holt stock-recruitment relationship. With this, we provide numerical estimates for one Chilean fishery.

4.1. *An age class dynamical model*

We consider an age-structured abundance population model with a possibly non linear stock-recruitment relationship, derived from fish stock management (see [6], and also [3] for more details).

Time is measured in years, and the time index $t \in \mathbb{N}$ represents the beginning of year t and of yearly period $[t, t+1[$. Let $A \in \mathbb{N}^*$ denote a maximum age, and $a \in \{1, \ldots, A\}$ an age class index, all expressed in years. The state is the vector $N = (N_a)_{a=1,\ldots,A} \in \mathbb{R}_+^A$, the *abundances* at age: for $a = 1, \ldots, A-1$, $N_a(t)$ is the number of individuals of age between $a-1$ and a at the beginning of yearly period $[t, t+1[$; $N_A(t)$ is the number of individuals of age greater than $A-1$. The control $u(t)$ is the *fishing effort (multiplier)*, supposed to be applied in the middle of period $[t, t+1[$. The control dynamical model is

$$N(t+1) = g\big(N(t), u(t)\big), \quad t = t_0, t_0+1, \ldots, \qquad N(t_0) = N_0 \text{ given},$$

where the vector function $g = (g_a)_{a=1,\ldots,A}$ is defined for any $N \in \mathbb{R}_+^A$ and $u \in \mathbb{R}_+$ by

$$\begin{cases} g_1(N,u) &= \varphi\big(SSB(N)\big)\,, \\ g_a(N,u) &= e^{-(M_{a-1}+uF_{a-1})}N_{a-1}, \quad a = 2, \ldots, A-1\,, \\ g_A(N,u) &= e^{-(M_{A-1}+uF_{A-1})}N_{A-1} + e^{-(M_A+uF_A)}N_A\,. \end{cases} \tag{13}$$

In the above formulas, M_a is the natural *mortality rate* of individuals of age a and F_a is the mortality rate of individuals of age a due to harvesting between t and $t+1$, supposed to remain constant during period $[t, t+1[$ (the vector $(F_a)_{a=1,\ldots,A}$ is termed the *exploitation pattern*). The function φ describes a *stock-recruitment relationship.* The *spawning stock biomass SSB* is defined by

$$SSB(N) := \sum_{a=1}^{A} \gamma_a w_a N_a\,, \tag{14}$$

that is summing the contributions of individuals to reproduction, where $(\gamma_a)_{a=1,\ldots,A}$ are the *proportions of mature individuals* (some may be zero) at age and $(w_a)_{a=1,\ldots,A}$ are the *weights at age* (all positive).

We will assume that this function is given by $\varphi(B) = \frac{B}{\alpha+\beta B}$ (allowing the cases $\alpha = 0$ or $\beta = 0$), known in the literature as the Beverton-Holt stock-recruitment relationship.

4.2. *Indicators reflecting conflicting preservation and production objectives*

We shall consider an acceptable set which reflects conflicting objectives of *preservation* – measured by the spawning stock biomass being high enough – and of *production*, measured by following yield indicator described below.

The production in term of biomass at the beginning of period $[t, t+1[$ is (see [6])

$$Y\big(N, u\big) = \sum_{a=1}^{A} w_a \, \frac{uF_a}{uF_a + M_a} \left(1 - e^{-(M_a + uF_a)}\right) N_a \, . \tag{15}$$

We focus our analysis on the acceptable set

$$\mathbb{D}_{(B_{\mathsf{lim}}, y_{\mathsf{min}})} := \{(N, u) \mid SSB(N) \geq B_{\mathsf{lim}}, \ Y(N, u) \geq y_{\mathsf{min}}\} \, , \tag{16}$$

where the yield function Y is given by (15) and SSB by (14).

Thus, our goal is, given an initial vector of abundance N_0, to compute the set of sustainable spawning stock biomass and catches given by

$$\mathcal{S}(N_0) := \left\{ \theta = (B_{\mathsf{lim}}, y_{\mathsf{min}}) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} \exists \; (u(t_0), u(t_0+1), \ldots) \text{ and} \\ (N(t_0), N(t_0+1), \ldots) \\ \text{satisfying } N(t_0) = N_0 \\ N(t+1) = g\big(N(t), u(t)\big) \\ \forall \; t = t_0, t_0+1, \ldots \text{ and} \\ SSB(N(t)) \geq B_{\mathsf{lim}} \\ Y(N(t), u(t)) \geq y_{\mathsf{min}} \end{array} \right\} . \tag{17}$$

It is straightforward to check that the dynamics g given by (13) and the indicators SSB and Y satisfy the property $\texttt{MONDAI}_k$ (see Definition 3.1) for $k = 1$.

4.3. *Numerical applications to the Chilean sea bass*

We provide numerical estimates of the set $\mathcal{S}(N_0)$ obtained for the species Chilean sea bass (*Dissostichus eleginoides*), harvested in the south of Chile. The dynamic of the Chilean sea bass can be described by the model (13) with a Beverton-Holt stock-recruitment relationship φ. The mortality is supposed to be the same at all ages. Numerical data have been provided by the *Centro de Estudios Pesqueros - Chile (CEPES).*

For an initial vector of abundances N_0, we have computed the set $\widehat{\mathcal{S}}(N_0)$ which, from Proposition 3.2, allows to determine the set $\mathcal{S}(N_0)$ of sustainable thresholds $(B_{\mathsf{lim}}, y_{\mathsf{min}})$.

The industrial harvesting of the Chilean sea bass started at 1988. The spawning stock biomass *SSB* with its respectively threshold B_{lim} are measured as the fraction of the *SSB* present at 1988, noted by SSB_0. A precautionary approach, indicated for this species, is to impose B_{lim} around $0.4SSB_0$. The productive thresholds, representing minimal catches, are measured in thousand of tons. The Figure 2 shows an schematic example: the curve in that case represents the set $\widehat{\mathcal{S}}(N_0)$ and, therefore, all the points below this curve are sustainable thresholds $(B_{\mathsf{lim}}, y_{\mathsf{min}})$ in $\mathcal{S}(N_0)$.

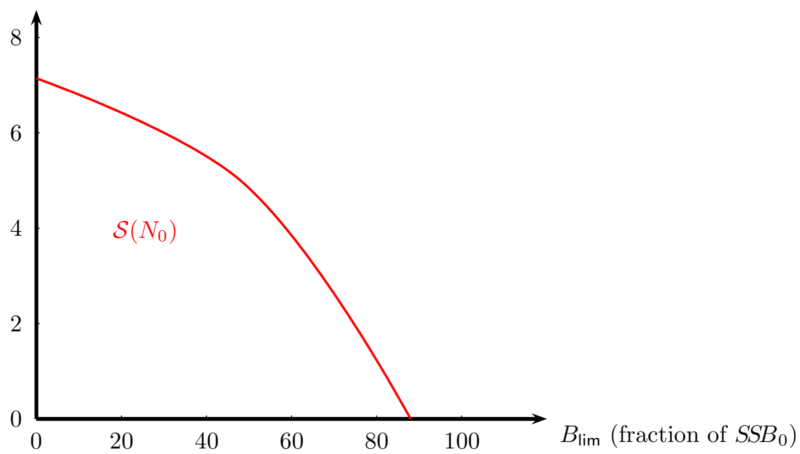

Figure 2. Set of sustainable thresholds $\mathcal{S}(N_0)$.

The last three figures show the sustainable thresholds $(B_{\mathsf{lim}}, y_{\mathsf{min}})$ considering three different initial states. We compute $\mathcal{S}(N_0)$ for N_0 the vector of abundances in the years 1988, 1997, and 2006.

In the Figure 3 we can see that with the vector of the abundance in

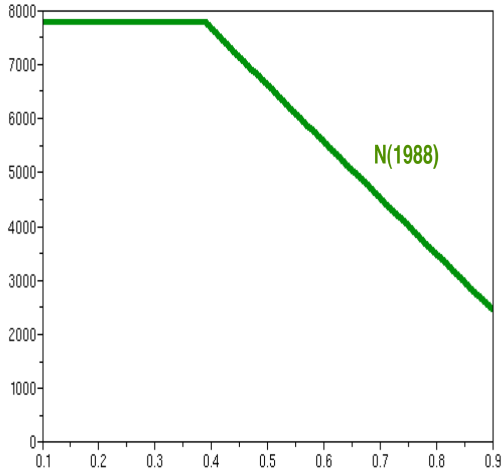

Figure 3. Set $\mathcal{S}(N_0)$ for N_0 the vector of abundances at year 1988.

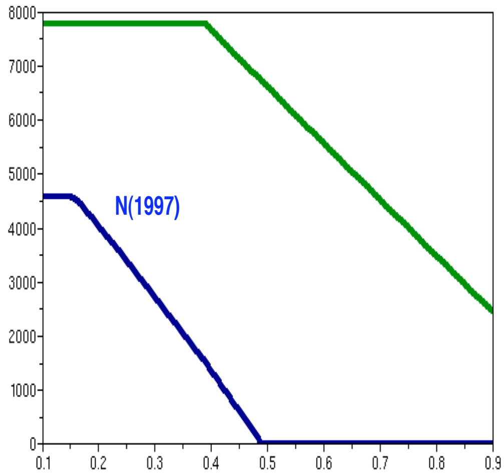

Figure 4. Set $\mathcal{S}(N_0)$ for N_0 the vector of abundances at year 1988 and 1997.

1988, for obtaining a spawning stock biomass above 0.4 SSB_0 we can assure at least 7 000 tons as minimal catches. Nevertheless, if we see the set of sustainable thresholds for the vector of abundances at year 2006, for obtaining the same threshold for SSB, we can not assure a positive minimal

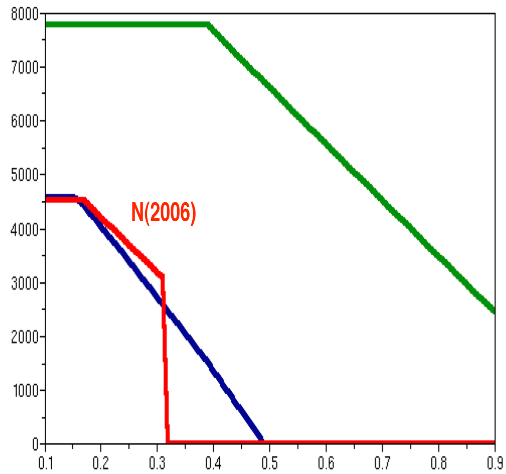

Figure 5. Set $\mathcal{S}(N_0)$ for N_0 the vector of abundances at year 1988, 1997, and 2006.

level of catches for the future year.

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